

# A Survey of Paraconsistent Logics

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**Abstract.** A survey of paraconsistent logics that are prominent representatives of the different approaches that have been followed to develop paraconsistent logics is provided. The paraconsistent logics that will be discussed are an enrichment of Priest's logic LP, the logic RM<sub>3</sub> from the school of relevance logic, da Costa's logics  $C_n$ , Jaśkowski's logic D<sub>2</sub>, and Subrahmanian's logics  $P\tau$ . A deontic logic based on the first of these logics will be discussed as well. Moreover, some proposed adaptations of the AGM theory of belief revision to paraconsistent logics will be mentioned.

## 1 Introduction

Paraconsistent logics are those logics that do not have the property that any formula can be deduced from every set of hypotheses that contains contradictory formulas. The paraconsistent logics that have been proposed differ in many ways. The differences are mostly minor, but occasionally major. Whether one paraconsistent logic is more plausible than another is fairly difficult to make out.

A logic with the property that any formula can be deduced from every set of hypotheses that contains contradictory formulas but one is far from a reasonable paraconsistent logic. Such a logic is in a certain sense a minimal paraconsistent logic. In a quest for paraconsistent logics that are maximally paraconsistent, many different paraconsistent logics have been proposed. Some of them are in fact maximally paraconsistent in a well-defined sense. However, there are other properties than maximal paraconsistency that are usually considered characteristic of reasonable paraconsistent logics. Among them is the property that the logic concerned does not validate deductions forbidden by classical logic that are not essential for paraconsistency. There are also less technical properties that are sometimes considered important. Among them are the ease with which the axiom schemas and inference rules of the logic concerned can be memorized and the ease with which the semantics of the logic concerned can be memorized.

As a rule, survey articles and handbook chapters on the subject of paraconsistent logic concentrate on explaining the nature of and motivation for the subject, giving a history of the subject, and/or surveying the basic techniques used to develop paraconsistent logics (see e.g. [49,52]). Individual paraconsistent logics proposed in the scientific literature are only touched to illustrate the techniques explained. The exceptions where a survey of several paraconsistent logics is provided, concentrate on discussing logics that have been developed following

a particular approach (see e.g. [14,30]). Gaining a basic understanding of a number of different paraconsistent logics and their interrelationships still requires an extensive study of scientific publications which contain many theoretical details that are not relevant to a basic understanding.

In this note, a survey of paraconsistent logics that are prominent representatives of the different approaches that have been followed to develop paraconsistent logics is provided. The survey is made so as to allow for gaining a basic understanding of the logics in question and their interrelationships. For each approach, the logic that has been selected as the prominent representative is one of the representatives for which the number of publications about work on it and/or the number of publications that cite the publications about work on it is relatively high. In a strict sense, the survey covers only a rather narrow group of paraconsistent logics. However, it is not so narrow as it seems at first, because the selected representative for each approach is in general closely related to most other representatives of the approach concerned.

The different approaches that have been followed to develop paraconsistent logics are:

- the *three-valued approach*: classical logic is turned into a logic based on three truth values: true, false and both-true-and-false;
- the *relevance approach*: classical logic is adapted to the idea that the antecedent of an implication must be relevant to its consequent;
- the *non-truth-functional approach*: classical logic is turned into a logic based on a non-truth-functional version of negation;
- the *non-adjunctive approach*: classical logic is adapted to the idea that the inference of  $A \wedge B$  from  $A$  and  $B$  must fail;
- the *annotation approach*: classical logic is changed into a logic where atomic formulas are annotated with believed truth values.

The representative of these approaches that will be discussed in this note are  $LP^\supset$ , which is Priest’s logic LP [48] enriched with an implication connective for which the standard deduction theorem holds, the logic  $RM_3$  [5] from the school of relevance logic, da Costa’s logics  $C_n$  [26], Jaśkowski’s logic  $D_2$  [40,39], and Subrahmanian’s logics  $P\tau$  [31], respectively.

Only propositional logics will be discussed. Details about the extensions of  $LP^\supset$ ,  $RM_3$ ,  $C_n$  and  $P\tau$  to the corresponding predicate logics can be found in [52], [35], [30] and [3], respectively.<sup>1</sup> These extensions are exactly as to be expected if universal quantification and existential quantification are regarded as generalized conjunction and generalized disjunction, respectively. Therefore, their discussion adds little to a basic understanding.

The survey of paraconsistent logics provided in this note stems from interest in modelling legal reasoning. However, legal reasoning is not only reasoning in

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<sup>1</sup> In [30], reference is made to several publications in which a formulation of  $D_2$  can be found. All formulations concerned, as well as the one in [43], are incorrect (cf. [24]). Consequently, the formulations of the extension of  $D_2$  to a predicate logic that can be found in some of these publications are incorrect as well.

the presence of inconsistent information, but also reasoning concerned with normative expressions such as obligatory, permissible, and prohibited. Therefore, a deontic logic based on one of the paraconsistent logics discussed in the survey,  $LP^\supset$ , is discussed in this note as well. Moreover, it is reasonable to assume that legal reasoning is done by rational persons who revise their beliefs in the light of new information. For that reason, proposed adaptations of the main theory of belief revision, the AGM theory, to paraconsistent logics and other work relevant to belief revision in the presence of inconsistent beliefs are also mentioned.

## 2 Preliminaries

Each logic that will be discussed in this note has the following logical connectives: an implication connective  $\supset$ , a conjunction connective  $\wedge$ , a disjunction connective  $\vee$ , and a negation connective  $\neg$ .<sup>2</sup> Bi-implication is in all cases defined as an abbreviation:  $A \equiv B$  stands for  $(A \supset B) \wedge (B \supset A)$ .

In  $LP^\supset$ ,  $RM_3$ ,  $C_n$  and  $D_2$ , the formation rules for formulas are the same as in classical propositional logic. Hence, the languages of  $LP^\supset$ ,  $RM_3$ ,  $C_n$  and  $D_2$  are identical. In  $P\tau$ , the formation rules for formulas differ in that the atomic formulas are annotated propositional variables instead of propositional variables.

For each logic that will be discussed in this note, a Hilbert-style formulation will be given. In those formulations,  $A$ ,  $B$  and  $C$  will be used as meta-variables ranging over all formulas of the logic concerned.

In the case of a Hilbert-style formulation, a proof of a formula  $A$  from a set of formulas  $\Gamma$  in a logic  $L$  is a sequence of formulas ending with  $A$  such that each formula in the sequence is either an axiom, or a formula in  $\Gamma$ , or a formula that follows from previous formulas in the sequence by one of the rules of inference.

The logical consequence relation of a logic  $L$ , denoted by  $\vdash_L$ , is the binary relation between sets of formulas and formulas defined as follows:  $\Gamma \vdash_L A$  iff there exists a proof of  $A$  from  $\Gamma$  in  $L$ .

A logic  $L$  is called a *paraconsistent* logic if its logical consequence relation  $\vdash_L$  satisfies the condition that there exist sets  $\Gamma$  of formulas of  $L$  and formulas  $A$  of  $L$  such that not for all formulas  $B$  of  $L$ ,  $\Gamma \cup \{A, \neg A\} \vdash_L B$ .

## 3 Priest's Paraconsistent Logic $LP^\supset$

In [48], Priest proposes the paraconsistent propositional logic LP (Logic of Paradox). The logic  $LP^\supset$  introduced in this section is LP enriched with an implication connective for which the standard deduction theorem holds. This logic is also known under the following names: PAC [16],  $PI^s$  [20] and pure CLuNs [21]. The fragment without the implication connective was already suggested by Asenjo in 1966 (see [15]).

A Hilbert-style formulation of  $LP^\supset$  is given in Table 1. This formulation is

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<sup>2</sup> Each of the connectives is requisite in the formulation of a representative of at least one of the approaches mentioned in Section 1.

**Table 1.** Hilbert-style formulation of  $LP^\supset$

<b>Axiom Schemas :</b>	
$A \supset (B \supset A)$	$\neg\neg A \equiv A$
$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	$\neg(A \supset B) \equiv A \wedge \neg B$
$((A \supset B) \supset A) \supset A$	$\neg(A \wedge B) \equiv \neg A \vee \neg B$
$(A \wedge B) \supset A$	$\neg(A \vee B) \equiv \neg A \wedge \neg B$
$(A \wedge B) \supset B$	
$A \supset (B \supset (A \wedge B))$	$A \vee \neg A$
$A \supset (A \vee B)$	
$B \supset (A \vee B)$	
$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$	
<b>Rule of Inference :</b>	
$\frac{A \quad A \supset B}{B}$	

taken from [16]. The axiom schemas on the left-hand side of Table 1 and the single inference rule (modus ponens) constitute a Hilbert-style formulation of the positive fragment of classical propositional logic. The first four axiom schemas on the right-hand side of Table 1 allow for the negation connective to be moved inward. We get a sound and complete formulation of the propositional part of the paraconsistent logic  $N^-$ , which was proposed by Nelson in [47], if we add these axiom schemas to a Hilbert-style formulation of the positive fragment of intuitionistic propositional logic. The fifth axiom schema on the right-hand side of Table 1 is the law of the excluded middle. This axiom schema can be thought of as saying that, for every proposition, the proposition or its negation is true, while leaving open the possibility that both are true. If we add the axiom schema  $\neg A \supset (A \supset B)$ , which says that any proposition follows from a contradiction, to the given Hilbert-style formulation of  $LP^\supset$ , then we get a Hilbert-style formulation of classical propositional logic (see e.g. [16]).

The following outline of the semantics of  $LP^\supset$  is based on [16]. Like in the case of classical propositional logic, meanings are assigned to the formulas of  $LP^\supset$  by means of valuations. However, in addition to the two classical truth values **t** (true) and **f** (false), a third meaning **b** (both true and false) may be assigned.

A *valuation* for  $LP^\supset$  is a function  $\nu$  from the set of all formulas of  $LP^\supset$  to the set  $\{\mathbf{t}, \mathbf{f}, \mathbf{b}\}$  such that for all formulas  $A$  and  $B$  of  $LP^\supset$ :

$$\nu(A \supset B) = \begin{cases} \mathbf{t} & \text{if } \nu(A) = \mathbf{f} \\ \nu(B) & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
\nu(A \wedge B) &= \begin{cases} \mathbf{t} & \text{if } \nu(A) = \mathbf{t} \text{ and } \nu(B) = \mathbf{t} \\ \mathbf{f} & \text{if } \nu(A) = \mathbf{f} \text{ or } \nu(B) = \mathbf{f} \\ \mathbf{b} & \text{otherwise,} \end{cases} \\
\nu(A \vee B) &= \begin{cases} \mathbf{t} & \text{if } \nu(A) = \mathbf{t} \text{ or } \nu(B) = \mathbf{t} \\ \mathbf{f} & \text{if } \nu(A) = \mathbf{f} \text{ and } \nu(B) = \mathbf{f} \\ \mathbf{b} & \text{otherwise,} \end{cases} \\
\nu(\neg A) &= \begin{cases} \mathbf{t} & \text{if } \nu(A) = \mathbf{f} \\ \mathbf{f} & \text{if } \nu(A) = \mathbf{t} \\ \mathbf{b} & \text{otherwise.} \end{cases}
\end{aligned}$$

The classical truth-conditions and falsehood-conditions for the logical connectives are retained. Except for implications, a formula is classified as both-true-and-false exactly when it cannot be classified as true or false by the classical truth-conditions and falsehood-conditions.

The logical consequence relation of  $\text{LP}^\supset$  is based on the idea that a valuation  $\nu$  satisfies a formula  $A$  if  $\nu(A) \in \{\mathbf{t}, \mathbf{b}\}$ . We have that  $\Gamma \vdash_{\text{LP}^\supset} A$  iff for every valuation  $\nu$ , either  $\nu(A') = \mathbf{f}$  for some  $A' \in \Gamma$  or  $\nu(A) \in \{\mathbf{t}, \mathbf{b}\}$ .

The following properties of  $\text{LP}^\supset$ , shown in [12], suggest that  $\text{LP}^\supset$  retains as much of classical propositional logic as possible:

- *containment in classical logic*: the logical consequence relation of  $\text{LP}^\supset$  is included in the logical consequence relation of classical propositional logic;
- *proper implication*: the implication connective of  $\text{LP}^\supset$  is such that the classical deduction theorem holds;
- *maximal paraconsistency relative to classical logic*: if we add a tautology of classical propositional logic that is not a theorem of  $\text{LP}^\supset$  as an axiom to  $\text{LP}^\supset$ , then we get a logic whose set of theorems is the set of all tautologies of classical propositional logic;
- *absolute strongly maximal paraconsistency*: any proper extension of the logical consequence relation of  $\text{LP}^\supset$  is not paraconsistent.

These properties make  $\text{LP}^\supset$  an ideal paraconsistent logic in the sense made precise in [12].

We get the following logics if we enrich  $\text{LP}^\supset$  with a constant for  $\mathbf{f}$ , with a constant for  $\mathbf{b}$  or with constants for both:

- $\text{LP}^{\supset, \mathbf{F}}$  is  $\text{LP}^\supset$  extended with a constant  $\mathbf{F}$  and the axiom schema  $\mathbf{F} \supset A$ . Valuations  $\nu$  for  $\text{LP}^{\supset, \mathbf{F}}$  are such that  $\nu(\mathbf{F}) = \mathbf{f}$ .
- $\text{LP}^{\supset, \mathbf{B}}$  is  $\text{LP}^\supset$  extended with a constant  $\mathbf{B}$  and the axiom schemas  $A \supset \mathbf{B}$  and  $A \supset \neg \mathbf{B}$ . Valuations  $\nu$  for  $\text{LP}^{\supset, \mathbf{B}}$  are such that  $\nu(\mathbf{B}) = \mathbf{b}$ .
- $\text{LP}^{\supset, \mathbf{F}, \mathbf{B}}$  is  $\text{LP}^\supset$  extended with constants  $\mathbf{F}$  and  $\mathbf{B}$  and the axiom schemas  $\mathbf{F} \supset A$ ,  $A \supset \mathbf{B}$  and  $A \supset \neg \mathbf{B}$ . Valuations  $\nu$  for  $\text{LP}^{\supset, \mathbf{F}, \mathbf{B}}$  are such that  $\nu(\mathbf{F}) = \mathbf{f}$  and  $\nu(\mathbf{B}) = \mathbf{b}$ .

The enrichments in question result in increase of expressive power. Below, the properties of  $\{\mathbf{t}, \mathbf{f}\}$ -closure and  $\{\mathbf{b}\}$ -freeness are used to characterize the expressive power of the different logics:

- a function  $g$  from  $\{t, f, b\}^n$  to  $\{t, f, b\}$  is  $\{t, f\}$ -closed if the image of the restriction of  $g$  to  $\{t, f\}^n$  is  $\{t\}$ ,  $\{f\}$  or  $\{t, f\}$ ;
- a function  $g$  from  $\{t, f, b\}^n$  to  $\{t, f, b\}$  is  $\{b\}$ -free if the image of the restriction of  $g$  to  $\{b\}^n$  is  $\{b\}$ .

The expressive power of  $LP^\supset$ ,  $LP^{\supset, F}$ ,  $LP^{\supset, B}$  and  $LP^{\supset, F, B}$  can now be characterized as follows:

- a function  $g$  from  $\{t, f, b\}^n$  to  $\{t, f, b\}$  is representable in the language of  $LP^\supset$  iff it is  $\{t, f\}$ -closed and  $\{b\}$ -free;
- a function  $g$  from  $\{t, f, b\}^n$  to  $\{t, f, b\}$  is representable in the language of  $LP^{\supset, F}$  iff it is  $\{t, f\}$ -closed;
- a function  $g$  from  $\{t, f, b\}^n$  to  $\{t, f, b\}$  is representable in the language of  $LP^{\supset, B}$  iff it is  $\{b\}$ -free;
- every function  $g$  from  $\{t, f, b\}^n$  to  $\{t, f, b\}$  is representable in the language of  $LP^{\supset, F, B}$ , i.e. the language of  $LP^{\supset, F, B}$  is functionally complete.

With the exception of  $\vee$  and  $\wedge$ , each of the connectives in  $\{\neg, \vee, \wedge, \supset, F, B\}$  is not definable in terms of the rest. The connectives  $\vee$  and  $\wedge$  are definable in terms of  $\{\neg, \supset, F, B\}$ . The preceding discussion of the expressive power of  $LP^\supset$  and some enrichments thereof is based on [17].

Note that, in  $LP^{\supset, F}$  and  $LP^{\supset, F, B}$ , a constant  $T$  for  $t$  can simply be defined by  $T = \neg F$ . Note further that the consistency of a formula  $A$  cannot be represented in  $LP^\supset$  and  $LP^{\supset, B}$ , but that it can be represented in  $LP^{\supset, F}$  and  $LP^{\supset, F, B}$  by the formula  $(A \supset F) \vee (\neg A \supset F)$ .<sup>3</sup> The properties that make the logic  $LP^\supset$  an ideal paraconsistent logic in the sense of [12] carry over to  $LP^{\supset, F}$ ,  $LP^{\supset, B}$  and  $LP^{\supset, F, B}$  (see e.g. [12]).  $LP^{\supset, F}$  is essentially the same logic as  $J_3$  [33] (see e.g. [23]).

## 4 Interlude: $LP^\supset$ and its Dual

It was mentioned in Section 3 that, if we add the axiom schema  $\neg A \supset (A \supset B)$  to the given Hilbert-style formulation of  $LP^\supset$ , then we get a Hilbert-style formulation of classical propositional logic. If we replace the axiom schema  $A \vee \neg A$  by the axiom schema  $\neg A \supset (A \supset B)$  in the given Hilbert-style formulation of  $LP^\supset$  instead, then we get a Hilbert-style formulation of Kleene's strong three-valued logic [41] enriched with an implication connective for which the standard deduction theorem holds. We use  $K_3^\supset$  to denote this logic.  $K_3^\supset$  can be considered to be the dual of  $LP^\supset$ . All differences between these two logics can be traced to the fact that the third truth value  $b$  is interpreted as both true and false in  $LP^\supset$  and as neither true nor false in  $K_3^\supset$ .

Like in the case of  $LP^\supset$ , meanings are assigned to the formulas of  $K_3^\supset$  by means of valuations that are functions from the set of all formulas of  $K_3^\supset$  to the set  $\{t, f, b\}$ . The conditions that a valuation for  $K_3^\supset$  must satisfy differ from the conditions that a valuation for  $LP^\supset$  must satisfy only with respect to implication:

<sup>3</sup> In the setting of da Costa's logics  $C_n$ , which will be discussed in Section 6, the consistency of a formula is called the well-behavedness of a formula.

$$\nu(A \supset B) = \begin{cases} \nu(B) & \text{if } \nu(A) = \mathbf{t} \\ \mathbf{t} & \text{otherwise.} \end{cases}$$

The logical consequence relation of  $K_3^\supset$ , denoted by  $\vdash_{K_3^\supset}$ , is the binary relation between sets of formulas of  $K_3^\supset$  and formulas of  $K_3^\supset$  defined as usual:  $\Gamma \vdash_{K_3^\supset} A$  iff there exists a proof of  $A$  from  $\Gamma$  in  $K_3^\supset$ . We have that  $\Gamma \vdash_{K_3^\supset} A$  iff for every valuation  $\nu$ , either  $\nu(A') \in \{\mathbf{f}, \mathbf{b}\}$  for some  $A' \in \Gamma$  or  $\nu(A) = \mathbf{t}$ .

In [10], a logic with four truth values ( $\mathbf{t}$ ,  $\mathbf{f}$ ,  $\mathbf{b}$  and  $\mathbf{n}$ ), called  $BL_\supset$ , is proposed in which both  $LP^\supset$  and  $K_3^\supset$  can be simulated (see e.g. [11]). This means in the case of  $LP^\supset$  that  $\Gamma \vdash_{LP^\supset} A$  iff  $\Gamma \cup \{P_1 \vee \neg P_1, \dots, P_n \vee \neg P_n\} \vdash_{BL_\supset} A$ , where  $\{P_1, \dots, P_n\}$  is the set of all propositional variables in  $\Gamma \cup \{A\}$ .

## 5 The Relevance Logic $RM_3$

The three-valued relevance-mingle logic  $RM_3$  [5] is the strongest logic among the logics that have been proposed by the school of relevance logic. Relevance logics are based on the idea that the antecedent of an implication must be relevant to its consequent. Although the meaning of relevance is nowhere made precise, it is a characteristic feature of a relevance logic that propositions of the forms  $A \supset (B \supset A)$  and  $A \supset (\neg A \supset B)$  do not belong to its theorems. This means that every relevance logic is a paraconsistent logic.

A Hilbert-style formulation of  $RM_3$  is given in Table 2. If we remove the last

**Table 2.** Hilbert-style formulation of  $RM_3$

<b>Axiom Schemas :</b>	
$A \supset A$	$B \supset (A \vee B)$
$(A \supset B) \supset ((B \supset C) \supset (A \supset C))$	$((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$
$A \supset ((A \supset B) \supset B)$	$(A \wedge (B \vee C)) \supset ((A \wedge B) \vee (A \wedge C))$
$(A \supset (A \supset B)) \supset (A \supset B)$	$\neg \neg A \supset A$
$(A \wedge B) \supset A$	$(A \supset \neg B) \supset (B \supset \neg A)$
$(A \wedge B) \supset B$	
$((A \supset B) \wedge (A \supset C)) \supset (A \supset (B \wedge C))$	$A \supset (A \supset A)$
$A \supset (A \vee B)$	$A \vee (A \supset B)$
<b>Rules of Inference :</b>	
$\frac{A \quad A \supset B}{B}$	$\frac{A \quad B}{A \wedge B}$

two axiom schemas on the right-hand side of Table 2 from the given Hilbert-style formulation of  $RM_3$ , then we get a Hilbert-style formulation of the well-known

logic R [5]. The axiom schema  $A \supset (A \supset A)$  is known as the mingle axiom. If we add the axiom schema  $A \supset (B \supset A)$ , which generalizes the mingle axiom, to the given Hilbert-style formulation of  $\text{RM}_3$ , then we get a Hilbert-style formulation of classical propositional logic (see e.g. [34]).

The following outline of the semantics of  $\text{RM}_3$  is based on [16]. Like in the case of  $\text{LP}^\supset$ , meanings are assigned to the formulas of  $\text{RM}_3$  by means of valuations that are functions from the set of all formulas of  $\text{RM}_3$  to the set  $\{\mathbf{t}, \mathbf{f}, \mathbf{b}\}$ . The conditions that a valuation for  $\text{RM}_3$  must satisfy differ from the conditions that a valuation for  $\text{LP}^\supset$  must satisfy only with respect to implication:

$$\nu(A \supset B) = \begin{cases} \mathbf{b} & \text{if } \nu(A) = \mathbf{b} \text{ and } \nu(B) = \mathbf{b} \\ \mathbf{t} & \text{if } \nu(A) = \mathbf{f} \text{ or } \nu(B) = \mathbf{t} \\ \mathbf{f} & \text{otherwise.} \end{cases}$$

The logical consequence relation of  $\text{RM}_3$  is also based on the idea that a valuation  $\nu$  satisfies a formula  $A$  if  $\nu(A) \in \{\mathbf{t}, \mathbf{b}\}$ , but is adapted to the different kind of implication found in  $\text{RM}_3$ . We have that  $\Gamma \vdash_{\text{RM}_3} A$  iff for every valuation  $\nu$ , either  $\nu(A') = \mathbf{f}$  for some  $A' \in \Gamma$ , or  $\nu(A) = \mathbf{t}$ , or  $\nu(A') = \mathbf{b}$  for all  $A' \in \Gamma$  and  $\nu(A) = \mathbf{b}$ .

By results from [12,13], it is easy to see that  $\text{RM}_3$  has the four properties that make it an ideal paraconsistent logic in the sense of [12] as well.

The implication connective of  $\text{LP}^\supset$ , here written  $\supset_*$ , can be defined in  $\text{RM}_3$  by

$$A \supset_* B = B \vee (A \supset B)$$

and the other way round, the implication connective  $\supset$  of  $\text{RM}_3$  can be defined in  $\text{LP}^\supset$  by

$$A \supset B = (A \supset_* B) \wedge (\neg B \supset_* \neg A)$$

(see e.g. [16]). Hence,  $\text{RM}_3$  has the same expressive power as  $\text{LP}^\supset$ . To increase the expressive power,  $\text{RM}_3$  can be enriched with a constant for  $\mathbf{f}$  in the same way as  $\text{LP}^\supset$  is enriched with a constant for  $\mathbf{f}$  in Section 3. In the resulting logic, like in  $\text{LP}^{\supset, \mathbf{f}}$ , the consistency of a formula  $A$  can be represented by the formula  $(A \supset \mathbf{f}) \vee (\neg A \supset \mathbf{f})$ .

## 6 Da Costa's Paraconsistent Logics $C_n$

In [26], da Costa proposes the paraconsistent propositional logics  $C_n$ , for  $n > 0$ . In these logics, paraconsistency is obtained by adopting weak forms of negations. These forms of negation are, to a certain extent, duals of the weak form of negation found in intuitionistic logic: if something is false then its negation must be true, but if something is true then its negation may be true as well. In the case that something is true, further conditions are imposed, but they never have the effect that its negation must be true. In this way, the logics in question allow for contradictory formulas to be true.

A formula of the form  $A \wedge \neg A$  is a contradictory formula. If the formula  $\neg(A \wedge \neg A)$  is true, then  $A$  is called a well-behaved formula of degree 1; if in



addition the formula  $\neg(\neg(A \wedge \neg A) \wedge \neg\neg(A \wedge \neg A))$  is true, then  $A$  is called a well-behaved formula of degree 2; etc. We introduce abbreviations to express that a formula is a well-behaved formula of degree  $n$ , for  $n > 0$ . The abbreviations  $A^{(n)}$ , for  $n > 0$ , are recursively defined as follows:  $A^{(1)}$  stands for  $A^1$ ,  $A^{(n+1)}$  stands for  $A^{(n)} \wedge A^{n+1}$ , where the auxiliary abbreviations  $A^n$ , for  $n \geq 0$ , are recursively defined as follows:  $A^0$  stands for  $A$ ,  $A^{n+1}$  stands for  $\neg(A^n \wedge \neg A^n)$ .

A Hilbert-style formulation of  $C_n$  is given in Table 3. The axiom schemas

**Table 3.** Hilbert-style formulation of  $C_n$

<b>Axiom Schemas :</b>	
$A \supset (B \supset A)$	$A \vee \neg A$
$(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$	$\neg\neg A \supset A$
$(A \wedge B) \supset A$	$B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$
$(A \wedge B) \supset B$	$(A^{(n)} \wedge B^{(n)}) \supset (A \supset B)^{(n)}$
$A \supset (B \supset (A \wedge B))$	$(A^{(n)} \wedge B^{(n)}) \supset (A \wedge B)^{(n)}$
$A \supset (A \vee B)$	$(A^{(n)} \wedge B^{(n)}) \supset (A \vee B)^{(n)}$
$B \supset (A \vee B)$	
$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$	
<b>Rule of Inference :</b>	
$\frac{A \quad A \supset B}{B}$	

on the left-hand side of Table 3 and the single inference rule (modus ponens) constitute a Hilbert-style formulation of the positive fragment of intuitionistic propositional logic. The third axiom schema on the right-hand side expresses that the version of reductio ad absurdum that is found in a Hilbert-style formulation of full of intuitionistic propositional logic, viz.  $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$ , works provided that  $B$  is a well-behaved formula of degree  $n$ . The last three axiom schemas say that formulas composed of well-behaved formulas of degree  $n$  are well-behaved formulas of degree  $n$  as well. If we add the axiom schema  $\neg(A \wedge \neg A)$ , which is called the law of noncontradiction, to the given Hilbert-style formulation of  $C_n$ , then we get a Hilbert-style formulation of classical propositional logic (see e.g. [30]).

The following outline of the semantics of  $C_n$  is based on [27]. Like in the cases of  $LP^\supset$  and  $RM_3$ , meanings are assigned to the formulas of  $C_n$  by means of valuations. However, different from valuations for  $LP^\supset$  and  $RM_3$ , valuations for  $C_n$  are functions from the set of all formulas of  $C_n$  to the set  $\{t, f\}$  (like in the case of classical propositional logic). The conditions that a valuation for  $C_n$  must satisfy differ from the conditions that a valuation for classical propositional logic must satisfy only with respect to negation:

$$\begin{aligned}
\nu(\neg A) = \mathbf{t} & \quad \text{if } \nu(A) = \mathbf{f} \\
\nu(A) = \mathbf{t} & \quad \text{if } \nu(\neg\neg A) = \mathbf{t} \\
\nu(A) = \mathbf{f} & \quad \text{if } \nu(A^{(n)}) = \mathbf{t} \text{ and } \nu(A \supset B) = \mathbf{t} \text{ and } \nu(A \supset \neg B) = \mathbf{t} \\
\nu((A \supset B)^{(n)}) = \mathbf{t} & \text{ if } \nu(A^{(n)}) = \mathbf{t} \text{ and } \nu(B^{(n)}) = \mathbf{t} \\
\nu((A \wedge B)^{(n)}) = \mathbf{t} & \text{ if } \nu(A^{(n)}) = \mathbf{t} \text{ and } \nu(B^{(n)}) = \mathbf{t} \\
\nu((A \vee B)^{(n)}) = \mathbf{t} & \text{ if } \nu(A^{(n)}) = \mathbf{t} \text{ and } \nu(B^{(n)}) = \mathbf{t}.
\end{aligned}$$

With these unusual conditions, there exist valuations that assign the truth value  $\mathbf{t}$  to at least one contradictory formula. Clearly, these conditions are nothing else but semantic counterparts of the axiom schemas in which the negation connective occurs. Therefore, they do not help in gaining a better insight into the negation connective of  $C_n$ . At best, these conditions confirm the feeling that the negation connective of  $C_n$  is not really a contrary forming operator. By the unusual conditions, unlike the valuations for  $LP^\supset$  and  $RM_3$ , the valuations for  $C_n$  are not fully determined by the truth values that they assign to the propositional variables. That is, negation is made non-truth-functional in  $C_n$ .

Like the logical consequence relation of classical propositional logic, the logical consequence relation of  $C_n$  is based on the idea that a valuation  $\nu$  satisfies a formula  $A$  if  $\nu(A) = \mathbf{t}$ . We have that  $\Gamma \vdash_{C_n} A$  iff for every valuation  $\nu$ , either  $\nu(A') = \mathbf{f}$  for some  $A' \in \Gamma$  or  $\nu(A) = \mathbf{t}$ .

The logical consequence relation of  $C_n$  is included in the logical consequence relation of classical propositional logic and the implication connective of  $C_n$  is such that the classical deduction theorem holds (see e.g. [30]). However, maximal paraconsistency relative to classical logic and absolute strongly maximal paraconsistency are not properties of  $C_n$  (see e.g. [23,18]). Consequently,  $C_n$  is not an ideal paraconsistent logic in the sense of [12].

Although the negation connective of  $C_n$  is non-truth-functional, the truth-functional negation connective of classical propositional logic, here written  $\neg_*$ , can be defined in  $C_n$  by  $\neg_* A = \neg A \wedge A^{(n)}$  (see e.g. [27]).

Little is known about the connections between the logics  $C_n$  and the other paraconsistent logics discussed in this note. In [23], Carnielli and others introduce several classes of paraconsistent logics, including the class of LFIs (Logics of Formal Inconsistency) and the class of dC-systems. The class of dC-systems is a subclass of the class of LFIs. The logics  $C_n$  are dC-systems. The logic  $LP^\supset$  is not even an LFI, because connectives for consistency and inconsistency must be definable in an LFI. However, the logics  $LP^{\supset, \mathbf{F}}$  and  $LP^{\supset, \mathbf{F}, \mathbf{B}}$  are dC-systems. An interesting collection of thousands of dC-systems with maximal paraconsistency relative to classical logic and absolute strongly maximal paraconsistency are identified in [23]. Any of the dC-systems in question can be conservatively translated into  $LP^{\supset, \mathbf{F}}$  and  $LP^{\supset, \mathbf{F}, \mathbf{B}}$ .<sup>4</sup> However, because maximal paraconsistency relative to classical logic and absolute strongly maximal paraconsistency are not properties of them, the logics  $C_n$  do not belong to the collection.

<sup>4</sup> For a characterization of the dC-systems in question and a definition of conservative translation, see [23].

## 7 Jaśkowski's Paraconsistent Logic $D_2$

In [40,39], Jaśkowski proposes the paraconsistent logic  $D_2$ .<sup>5</sup> Jaśkowski calls this logic a discussive logic. His basic idea is to take true as true according to the position of some person engaged in a discussion. True in this sense can be thought of as true in some possible world, namely the world of some person's position. Thus, both  $A$  and  $\neg A$  can be true without an arbitrary formula  $B$  being true. Initially,  $D_2$  was presented as a modal logic in disguise.

A Hilbert-style formulation of  $D_2$  is given in Table 4. This formulation is

**Table 4.** Hilbert-style formulation of  $D_2$

<b>Axiom Schemas :</b>	
$A \supset (B \supset A)$	$\neg(\neg A \wedge \neg\neg A \wedge \neg(A \vee \neg A))$
$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	$\neg(\neg A \wedge \neg B \wedge \neg(A \vee B)) \supset$
$(A \wedge B) \supset A$	$\neg(\neg A \wedge \neg B \wedge \neg C \wedge \neg(A \vee B \vee C))$
$(A \wedge B) \supset B$	$\neg(\neg A \wedge \neg B \wedge \neg C \wedge \neg(A \vee B \vee C)) \supset$
$(A \supset B) \supset ((A \supset C) \supset (A \supset (B \wedge C)))$	$\neg(\neg A \wedge \neg C \wedge \neg B \wedge \neg(A \vee C \vee B))$
$A \supset (A \vee B)$	$\neg(\neg A \wedge \neg B \wedge \neg C \wedge \neg(A \vee B \vee C)) \supset$
$B \supset (A \vee B)$	$((A \vee B \vee \neg C) \supset (A \vee B))$
$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$	$\neg(\neg A \wedge \neg B) \supset (A \vee B)$
$A \vee (A \supset B)$	$(A \vee (B \vee \neg B)) \supset \neg(\neg A \wedge \neg(B \vee \neg B))$
<b>Rule of Inference :</b>	
$\frac{A \quad A \supset B}{B}$	

taken from [24]. Various axiom schemas from Table 4 are rather unusual as axiom schemas. However, they are all schemas of tautologies of classical propositional logic.

The following outline of the semantics of  $D_2$  is based on [24]. Unlike in the cases of  $LP^\supset$ ,  $RM_3$  and  $C_n$ , meanings are assigned to the formulas of  $D_2$  by means of pairs  $(W, \nu)$  where  $W$  is a non-empty set of *worlds* and  $\nu$  is a function from the cartesian product of the set of all formulas of  $D_2$  and the set  $W$  to the set  $\{t, f\}$  such that for all formulas  $A$  and  $B$  of  $D_2$ :

$$\begin{aligned} \nu(A \supset B, w) &= \begin{cases} t & \text{if for all } w' \in W, \nu(A, w') = f \text{ or } \nu(B, w) = t \\ f & \text{if for some } w' \in W, \nu(A, w') = t \text{ and } \nu(B, w) = f, \end{cases} \\ \nu(A \wedge B, w) &= \begin{cases} t & \text{if } \nu(A, w) = t \text{ and for some } w' \in W, \nu(B, w') = t \\ f & \text{if } \nu(A, w) = f \text{ or for all } w' \in W, \nu(B, w') = f, \end{cases} \end{aligned}$$

<sup>5</sup> The cited publications are translations of Polish editions published in 1948 and 1949.

$$\begin{aligned}\nu(A \vee B, w) &= \begin{cases} \mathbf{t} & \text{if } \nu(A, w) = \mathbf{t} \text{ or } \nu(B, w) = \mathbf{t} \\ \mathbf{f} & \text{if } \nu(A, w) = \mathbf{f} \text{ and } \nu(B, w) = \mathbf{f}, \end{cases} \\ \nu(\neg A, w) &= \begin{cases} \mathbf{t} & \text{if } \nu(A, w) = \mathbf{f} \\ \mathbf{f} & \text{if } \nu(A, w) = \mathbf{t}. \end{cases}\end{aligned}$$

These pairs are called discussive structures. A discussive structure is essentially the same as a Kripke structure of which the accessibility relation includes every pair of worlds. The truth-conditions and falsehood-conditions for the logical connectives become the classical ones if the number of worlds is restricted to one. The conditions for the implication connective and the conjunction connective reveal their modal nature clearly.

The logical consequence relation of  $D_2$  is based on the idea that a discussive structure  $(W, \nu)$  satisfies a formula  $A$  if  $\nu(A, w) = \mathbf{t}$  for some  $w \in W$ . We have that  $\Gamma \vdash_{D_2} A$  iff for every discussive structure  $(W, \nu)$ , either  $\nu(A', w') = \mathbf{f}$  for all  $w' \in W$  for some  $A' \in \Gamma$  or  $\nu(A, w) = \mathbf{t}$  for some  $w \in W$ .

The logical consequence relation of  $D_2$  is included in the logical consequence relation of classical propositional logic and the implication connective of  $D_2$  is such that the classical deduction theorem holds (see e.g. [24]). It is unknown to me whether maximal paraconsistency relative to classical logic and/or absolute strongly maximal paraconsistency are properties of  $D_2$  and consequently whether  $D_2$  is an ideal paraconsistent logic in the sense of [12].

Little is known about the connections between the logic  $D_2$  and the other paraconsistent logics discussed in this note. Like  $LP^\supset$ ,  $D_2$  is not even an LFI. If  $D_2$  is enriched with the necessity operator  $\Box$  satisfying the axioms of S5, which is the modal operator used in [40,39] to explain  $D_2$ , we get an LFI.

## 8 The Annotated Logics $P\tau$

In [54], Subrahmanian takes the first step towards the paraconsistent propositional logics called  $P\tau$  [31]. Here,  $\tau$  is some triple  $(|\tau|, \leq, \sim)$ , where  $(|\tau|, \leq)$  is a complete lattice of (object-level) truth values and  $\sim$  is a function  $\sim: |\tau| \rightarrow |\tau|$  that gives the meaning of negation in  $P\tau$ . A typical case is the one where  $|\tau| = \{\mathbf{n}, \mathbf{t}, \mathbf{f}, \mathbf{b}\}$ ,  $\mathbf{n} \leq x$ ,  $x \leq \mathbf{b}$ ,  $x \not\leq \sim(x)$  if  $x \in \{\mathbf{t}, \mathbf{f}\}$ ,  $\sim(\mathbf{t}) = \mathbf{f}$ ,  $\sim(\mathbf{f}) = \mathbf{t}$ , and  $\sim(x) = x$  if  $x \in \{\mathbf{n}, \mathbf{b}\}$ .  $P\tau$  is called an annotated logic. In  $P\tau$ , propositional variables are annotated with an element from  $|\tau|$ . An annotated propositional variable  $P_\lambda$ , where  $P$  is an ordinary propositional variable and  $\lambda \in |\tau|$ , expresses that it is believed that  $P$ 's truth value is at least  $\lambda$ .

We use the symbols  $\perp$  and  $\top$  to denote the bottom element and the top element, respectively, of the complete lattice  $(|\tau|, \leq)$ . Moreover, we write  $\bigsqcup_{i=1}^n \lambda_i$ , where  $\lambda_1, \dots, \lambda_n \in |\tau|$ , for the least upper bound of the set  $\{\lambda_1, \dots, \lambda_n\}$  with respect to  $\leq$ .

We also introduce abbreviations for multiple negations. The abbreviations  $\neg^n A$ , for  $n \geq 0$ , are recursively defined as follows:  $\neg^0 A$  stands for  $A$  and  $\neg^{n+1} A$  stands for  $\neg(\neg^n A)$ . A formula which is not of the form  $\neg^n P_\lambda$ , where  $P_\lambda$  is an annotated propositional variable, is called a complex formula.

**Table 5.** Hilbert-style formulation of  $P\tau$

<b>Axiom Schemas :</b>	
$A \supset (B \supset A)$	$(F \supset G) \supset ((F \supset \neg G) \supset \neg F)$
$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	$F \supset (\neg F \supset A)$
$((A \supset B) \supset A) \supset A$	$F \vee \neg F$
$(A \wedge B) \supset A$	
$(A \wedge B) \supset B$	$P_{\perp}$
$A \supset (B \supset (A \wedge B))$	$\neg^{n+1} P_{\lambda} \equiv \neg^n P_{\sim(\lambda)}$
$A \supset (A \vee B)$	$P_{\lambda} \supset P_{\mu} \quad \text{if } \lambda \geq \mu$
$B \supset (A \vee B)$	$P_{\lambda_1} \wedge \dots \wedge P_{\lambda_n} \supset P_{\lambda} \quad \text{if } \lambda = \bigsqcup_{i=1}^n \lambda_i$
$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$	
<b>Rule of Inference :</b>	
$A \quad A \supset B$	
$B$	

A Hilbert-style formulation of  $P\tau$  is given in Table 5. In this table,  $F$  and  $G$  range over all complex formulas of  $P\tau$ ,  $P$  ranges over propositional variables, and  $\lambda, \mu, \lambda_1, \dots, \lambda_n$  range over  $|\tau|$ . This formulation is based on the formulation of  $Q\tau$ , the first-order counterpart of  $P\tau$ , given in [3]. The axiom schemas on the left-hand side of Table 5 and the single inference rule (modus ponens) constitute a Hilbert-style formulation of the positive fragment of classical propositional logic. The first three axiom schemas on the right-hand side of Table 5 are the usual axiom schemas for negation in a Hilbert-style formulation of classical propositional logic, but here their possible instances are restricted. The last four axiom schemas on the right-hand side of Table 5 are special axiom schemas concerning formulas of the forms  $P_{\lambda}$  and  $\neg^{n+1} P_{\lambda}$ .

The following outline of the semantics of  $P\tau$  is based on [3]. Like in the cases of  $LP^{\supset}$ ,  $RM_3$  and  $C_n$ , meanings are assigned to the formulas of  $P\tau$  by means of valuations. Like in the cases of  $C_n$ , valuations for  $P\tau$  are functions from the set of all formulas of  $P\tau$  to the set  $\{t, f\}$ . A valuation for  $P\tau$  is such that for all formulas  $A$  and  $B$  of  $P\tau$ , all complex formulas  $F$  of  $P\tau$ , and all propositional variables  $P$ :

$$\begin{aligned} \nu(A \supset B) &= \begin{cases} t & \text{if } \nu(A) = f \text{ or } \nu(B) = t \\ f & \text{if } \nu(A) = t \text{ and } \nu(B) = f, \end{cases} \\ \nu(A \wedge B) &= \begin{cases} t & \text{if } \nu(A) = t \text{ and } \nu(B) = t \\ f & \text{if } \nu(A) = f \text{ or } \nu(B) = f, \end{cases} \\ \nu(A \vee B) &= \begin{cases} t & \text{if } \nu(A) = t \text{ or } \nu(B) = t \\ f & \text{if } \nu(A) = f \text{ and } \nu(B) = f, \end{cases} \end{aligned}$$

$$\begin{aligned}
\nu(\neg F) &= \begin{cases} \mathbf{t} & \text{if } \nu(F) = \mathbf{f} \\ \mathbf{f} & \text{if } \nu(F) = \mathbf{t}, \end{cases} \\
\nu(\neg^{n+1} P_\lambda) &= \begin{cases} \mathbf{t} & \text{if } \nu(\neg^n P_{\sim(\lambda)}) = \mathbf{t} \\ \mathbf{f} & \text{if } \nu(\neg^n P_{\sim(\lambda)}) = \mathbf{f}, \end{cases} \\
\nu(P_\lambda) &= \begin{cases} \mathbf{t} & \text{if for some } \mu \geq \lambda, \nu(P_\mu) = \mathbf{t} \\ \mathbf{f} & \text{if for all } \mu \geq \lambda, \nu(P_\mu) = \mathbf{f}. \end{cases}
\end{aligned}$$

The classical truth-conditions and falsehood-conditions for the logical connectives are retained except for the negation connective. There are special conditions for the occurrences of the negation connective in formulas of the form  $\neg^{n+1} P_\lambda$ .

Like the logical consequence relation of  $C_n$ , the logical consequence relation of  $P\tau$  is based on the idea that a valuation  $\nu$  satisfies a formula  $A$  if  $\nu(A) = \mathbf{t}$ . We have that  $\Gamma \vdash_{P\tau} A$  iff for every valuation  $\nu$ , either  $\nu(A') = \mathbf{f}$  for some  $A' \in \Gamma$  or  $\nu(A) = \mathbf{t}$ .

The properties that make a paraconsistent logic ideal in the sense of [12] are unmeaning in the case of  $P\tau$ , because its language is substantially deviating due to the annotated propositional variables.

The negation connective of classical propositional logic, here written  $\neg_*$ , can be defined in  $P\tau$  by  $\neg_* A = A \supset ((A \supset A) \wedge \neg(A \supset A))$  (see e.g. [3]).

It seems that nothing is known about the connections between the logic  $P\tau$  and the other paraconsistent logics discussed in this note. This is not completely unexpected, because the language of  $P\tau$  is substantially deviating.

## 9 Paraconsistent Deontic Logics

Legal reasoning is reasoning in the presence of inconsistent information and reasoning concerned with normative expressions such as obligatory, permissible, and prohibited. This calls for a paraconsistent deontic logic. Because paraconsistent deontic logics have not been investigated extensively, only one is discussed in some detail here. The logic concerned is called  $DLP^{\supset, \mathbf{f}}$  because it is a deontic logic based on  $LP^{\supset, \mathbf{f}}$ . It is essentially the same logic as the deontic logic DLF11 proposed in [25].

$DLP^{\supset, \mathbf{f}}$  has the connectives of  $LP^{\supset, \mathbf{f}}$  and in addition an obligation connective  $\mathcal{O}$ . A formulation of this logic is obtained by adding to the Hilbert-style formulation of  $LP^{\supset}$  given in Table 1 the axiom schema  $\mathbf{F} \supset A$  and the deontic axiom schemas and rule of inference given in Table 6. Of course, the additional rule of inference does not belong in a genuine Hilbert-style formulation of a logic. Like in  $LP^{\supset}$ , the standard deduction theorem holds in  $DLP^{\supset, \mathbf{f}}$ .

Unlike in the case of  $LP^{\supset}$ , but like in the case of other modal logics, meanings are assigned to the formulas of  $DLP^{\supset, \mathbf{f}}$  by means of triples  $(W, R, \nu)$  where  $W$  is a non-empty set of *possible worlds*,  $R \subseteq W \times W$  is an *accessibility* relation for which it holds that for all  $w \in W$  there exists a  $w' \in W$  such that  $wRw'$ , and  $\nu$  is a function from the cartesian product of the set of all formulas of  $DLP^{\supset, \mathbf{f}}$  and the set  $W$  to the set  $\{\mathbf{t}, \mathbf{f}, \mathbf{b}\}$  such that for all formulas  $A$  and  $B$  of  $DLP^{\supset, \mathbf{f}}$ :

**Table 6.** Deontic axiom schemas and rule of inference for  $DLP^{\supset, F}$

<b>Axiom Schemas :</b>
$\mathcal{O}(A \supset B) \supset (\mathcal{O}A \supset \mathcal{O}B)$
$\mathcal{O}F \supset F$
<b>Rule of Inference :</b>
$\frac{A \text{ is a theorem}}{\mathcal{O}A \text{ is a theorem}}$

$$\begin{aligned}
\nu(A \supset B, w) &= \begin{cases} t & \text{if } \nu(A, w) = f \\ \nu(B, w) & \text{otherwise,} \end{cases} \\
\nu(A \wedge B, w) &= \begin{cases} t & \text{if } \nu(A, w) = t \text{ and } \nu(B, w) = t \\ f & \text{if } \nu(A, w) = f \text{ or } \nu(B, w) = f \\ b & \text{otherwise,} \end{cases} \\
\nu(A \vee B, w) &= \begin{cases} t & \text{if } \nu(A, w) = t \text{ or } \nu(B, w) = t \\ f & \text{if } \nu(A, w) = f \text{ and } \nu(B, w) = f \\ b & \text{otherwise,} \end{cases} \\
\nu(\neg A, w) &= \begin{cases} t & \text{if } \nu(A, w) = f \\ f & \text{if } \nu(A, w) = t \\ b & \text{otherwise,} \end{cases} \\
\nu(F, w) &= f, \\
\nu(\mathcal{O}A, w) &= \begin{cases} t & \text{if for all } w' \in W \text{ with } wRw', \nu(A, w') = t \\ f & \text{if for some } w' \in W \text{ with } wRw', \nu(A, w') = f \\ b & \text{otherwise.} \end{cases}
\end{aligned}$$

These triples are called three-valued Kripke structures. The truth-conditions and falsehood-conditions for the logical connectives  $\supset$ ,  $\wedge$ ,  $\vee$  and  $\neg$  are the ones used in the semantics of  $LP^{\supset}$  for each possible world. The conditions for the obligation connective reveals its modal nature clearly.

The logical consequence relation of  $DLP^{\supset, F}$  is based on the idea that a three-valued Kripke structure  $(W, R, \nu)$  satisfies a formula  $A$  if  $\nu(A, w) \in \{t, b\}$  for all  $w \in W$ . We have that  $\Gamma \vdash_{DLP^{\supset, F}} A$  iff for every three-valued Kripke structure  $(W, R, \nu)$  and  $w \in W$ , either  $\nu(A', w) = f$  for some  $A' \in \Gamma$  or  $\nu(A, w) \in \{t, b\}$ .

$DLP^{\supset, F}$  is a *deontically paraconsistent* logic, i.e. there exist sets  $\Gamma$  of formulas of  $DLP^{\supset, F}$  and formulas  $A$  of  $DLP^{\supset, F}$  such that not for all formulas  $B$  of  $DLP^{\supset, F}$ ,  $\Gamma \cup \{\mathcal{O}A, \mathcal{O}\neg A\} \vdash_{DLP^{\supset, F}} \mathcal{O}B$ . A formula  $A$  of  $DLP^{\supset, F}$  is called *deontically inconsistent* if  $\mathcal{O}\neg((A \supset F) \vee (\neg A \supset F))$ . From contradictory obligations  $\mathcal{O}A$  and  $\mathcal{O}\neg A$ , it can be inferred that  $A$  is deontically inconsistent: for all sets  $\Gamma$  of formulas of  $DLP^{\supset, F}$  and all formulas  $A$  of  $DLP^{\supset, F}$ , we have that  $\Gamma \vdash_{DLP^{\supset, F}} \mathcal{O}A \supset (\mathcal{O}\neg A \supset \mathcal{O}\neg((A \supset F) \vee (\neg A \supset F)))$ .

Deontic logics based on  $RM_3$  and  $C_1$  are devised in a similar way in [46] and [29], respectively.<sup>6</sup> Published work on deontic logics based on  $D_2$  and  $P\tau$  seems to be non-existent. As far as modal logics based on  $P\tau$  are concerned, all published work seems to be on epistemic logics (see e.g. [1]). In fact, virtually all published work on paraconsistent modal logics seems to be on epistemic logics based on  $P\tau$ .

## 10 Belief Revision and Related Issues

It is reasonable to assume that legal reasoning is done by rational persons who revise their beliefs in the light of new information. The AGM theory of belief revision [4,37] is a theory about the dynamics of the beliefs of a rational person that is based on the representation of beliefs as formulas of some logic. Most of the work on belief revision is based on the AGM theory. Much of this work takes for granted that the beliefs of a rational person must be consistent. However, motivated by the need to account for belief revision in the presence of inconsistent beliefs, some work has been done on the adaptation of the AGM theory to paraconsistent logics.

In [53], the AGM theory is adapted to a fragment of the logic  $BL_{\supset}$  mentioned at the end of Section 4 enriched with constants for  $t$  and  $f$  and it is shown that the adaptation has only minor consequences. A similar adaptation is found in [44]. In [36], it is shown how an adaptation of the AGM theory to the same fragment of  $BL_{\supset}$  can be obtained via a translation to classical logic. In [28], the AGM theory is adapted to the logics  $C_n$ . In [45], a new theory of belief revision for  $R$ , a relevance logic weaker than  $RM_3$ , is developed and it is sketched how this new theory is connected with the AGM theory. The new theory is also applicable to  $RM_3$ . A very general model of belief revision, in which all postulates of the AGM theory fail, is proposed in [51]. In [55], a survey of proposed adaptations of the AGM theory for non-classical logics, including paraconsistent logics, is provided.

In [7,8,9], a general framework is developed for reasoning where conclusions are drawn according to the most plausible interpretations,<sup>7</sup> i.e. the interpretations that are as close as possible to the set of hypotheses, and also as faithful as possible to the more reliable or important hypotheses in this set. This kind of reasoning emerges for example in belief revision with minimal change and in integration of information from different autonomous sources. It also encompasses adaptive reasoning, i.e. reasoning where, if a set of hypotheses can be split up into a consistent part and an inconsistent part, every assertion that is not related to the inconsistent part and classically follows from the consistent part, is deduced from the whole set. It is very likely that the kind of reasoning covered by the framework occurs in legal reasoning as well. The framework focusses on the consequence relations of logics for this kind of reasoning and it is

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<sup>6</sup> In fact, a family of modal logics based on  $RM_3$ , including a deontic logic is devised in [46].

<sup>7</sup> In the case of propositional logics, the interpretations are usually valuations.



based on two principles: a distance-based preference relation on interpretations and prioritized hypotheses.

A well-known paraconsistent logic in which a preference relation on interpretations underlies the consequence relation is LPm [50] (minimally inconsistent LP). In [42], variants of  $LP^{\supset, F, B}$  are studied in which a preference relation on interpretations underlies the consequence relation. A characteristic property of logics in which a preference relation on interpretations underlies the consequence relation is that they are paraconsistent, even if the point of departure is a non-paraconsistent logic. For the sake of completeness, we mention that the above-mentioned framework generalizes earlier work on preferential reasoning presented in [11]. The work on preferential reasoning presented in [19] seems to originate from that earlier work as well.

## 11 Concluding remarks

The discussions of the different paraconsistent logics included in the survey do not give known theoretical details about them that are not relevant to a basic understanding of them or their interrelationships. The details concerned can be found in the cited publications. The amount of detail that is given in this note differs from one logic to another for the simple reason that what is known about the questions concerned differs from one logic to another.

Not all paraconsistent logics discussed in this note have been investigated in an equally extensive way.  $LP^{\supset}$  has been investigated most extensively and  $D_2$  has been investigated least extensively. However,  $D_2$  is mentioned in virtually all publications on paraconsistent logics. It seems that the logics  $C_n$  are criticized most. It also seems that, with the exception of paraconsistent annotated logics, the applications of paraconsistent logics are only looked for and found in mathematics.

Paraconsistent annotated logics stem from logic programming in the presence of inconsistent information (see e.g. [22]). Other applications of paraconsistent annotated logics include database query answering [6], negotiation in multi-agent systems [38], expert system diagnosis [32] and robot control [2] in the presence of inconsistent information. In some of these applications, it is the case that  $|\tau| = [0, 1] \times [0, 1]$ . In those cases, the logic concerned has some characteristics of a fuzzy logic.

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